Table 1

N	β	I	$\|\delta y\|_{L_\infty} \leqslant$	$\|\delta y\|_{L_2} \leqslant$
0.5	0.157	0.0169	0.113	0.0533
1	0.236	0.03457	0.172	0.0766
2	0.329	0.329	0.25	0.108

Note that the inequality (12) could be 'optimized' by choosing such a value for $\varepsilon > 0$ that will make the estimate that follows from equation (12) minimal.

For completeness we cite the L_2 error bound derived in [3],

$$\|\delta y\|_{L_2} = \left(\int_0^1 \delta y^2 \, dx\right)^{1/2} = \left(\frac{1}{A_{\min}\lambda_1^4 + 2\lambda_1^2 + C_{\min}}\right)^{1/2}. \quad (18)$$

To illustrate the procedure, we performed calculations with the trial functions of the form

$$Y = 1 - \beta \sin\frac{\pi}{2}x \quad (19)$$

where β is a constant to be determined. Substituting equation (19) into (3) and minimizing with respect to β the values given in Table 1 are obtained. With these values for β, estimates for errors are calculated from equation (12) (with $\varepsilon = 0.1$) and equation (18).

Finally, we note that the accuracy of the approximate solution and the bounds on errors could be improved by taking more elaborate trial functions.

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A COMPARISON OF APPROXIMATION FORMULAS FOR THE HEAT CONDUCTION EQUATION OF A SPHERE

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NOMENCLATURE

- A_i see equation (12)
- Bi Biot number, hr_s/k
- C_i see equation (6)
- Fo Fourier number, dimensionless time, $\alpha t/r_s^2$
- Fo_δ Fo, when δ approaches 1
- f see equation (9)
- h heat-transfer coefficient [W m^{-2} K^{-1}]
- k thermal conductivity of the pellet [W m^{-1} K^{-1}]
- r dimensionless radial distance, referred to r_s
- r_s radius of spherical particle [m]
- t time [s]
- T dimensionless temperature, referred to T_0
- T_g dimensionless gas temperature, referred to T_0
- T_s dimensionless pellet surface temperature, referred to T_0
- $T_{s,\delta}$ T_s, when δ approaches 1
- T_0 initial temperature [K]

Greek symbols

- α thermal diffusivity of the pellet [m^2 s^{-1}]
- δ dimensionless penetration depth of the thermal wave, measured from the particle surface, referred to r_s
- ϑ dimensionless surface temperature, $(T_s - T_g)/(1 - T_g)$
- θ dimensionless quantity, $1 - (1 - Bi)\vartheta$

1. INTRODUCTION

THE PURPOSE of the present note is to compare approximate solutions of the heat conduction equation for a sphere found in the literature [1–3] and to give an improved approximation for long heating times. In particular, attention is focused on the surface temperature which plays an important role in many engineering problems, e.g. the convective heat transfer to particles or droplets in two-phase flow. As is well known, for many types of boundary conditions, there exist analytical solutions of the heat conduction equation expressed in terms of infinite series [4]. In more complicated cases numerical exact solutions can be obtained from a number of capable and flexible softwares [5]. However, within the scope of a complex two-phase flow problem which can only be treated numerically [1–3], one cannot employ the solution for single particles. This fact is essentially due to insufficient computer field length and too long computation time. Therefore, the use of approximate solutions is necessary. Moreover, the application of approximate methods is justified in practical engineering, where the general validity of a solution is not as important as a fast and simple treatment of the problem to be solved. For that reason, in the present paper a comparison between the approximation formulas mentioned above [1–3], a solution obtained under the assumption of a spatial uniform pellet temperature, and a numerical exact solution is presented. The validity range of the best approximation formula is extended to larger time intervals. The ODE's obtained from this approximation method are solved analytically for the case of constant heat transfer coefficient and constant gas temperature.

2. GOVERNING EQUATION AND METHOD OF APPROXIMATE SOLUTION

The basic equation is the heat conduction equation for a sphere,

$$\frac{\partial(rT)}{\partial Fo} = \frac{\partial^2(rT)}{\partial r^2} \quad (1)$$

subject to the initial and boundary conditions

$$T(0,r) = 1, \qquad 0 \leqslant r \leqslant 1, \tag{2a}$$

$$\frac{\partial T}{\partial r}(Fo,0) = 0, \qquad Fo \geqslant 0, \tag{2b}$$

$$\frac{\partial T}{\partial r}(Fo,1) = Bi(T_g - T_s), \qquad Fo > 0. \tag{2c}$$

Boundary condition (2c) is not transformed into a homogeneous form because T_g and Bi may depend upon Fo.

The ODE's describing the approximate solutions are obtained employing integral methods. Since the original concept of these methods is well documented in the literature [6], the method of solution is only briefly outlined here. It is assumed that $T(Fo,r)$ can be represented in the interval $[\delta(Fo), 1]$ by a polynomial in r with four (*a priori* unknown) coefficients only depending upon Fo. Initially, the effect of the boundary conditions at $r = 0$ is not felt and therefore, conditions (2a) and (2b) are replaced by the natural conditions,

$$T(Fo, 1-\delta) = 1, \quad 0 \leqslant Fo \leqslant Fo_\delta, \tag{3a}$$

$$\frac{\partial T}{\partial r}(Fo, 1-\delta) = 0, \quad 0 \leqslant Fo \leqslant Fo_\delta. \tag{3b}$$

Since four coefficients are to be determined an additional condition is derived

$$\frac{\partial^2 T}{\partial r^2}(Fo, 1-\delta) = 0, \quad 0 \leqslant Fo \leqslant Fo_\delta. \tag{3c}$$

After carrying out the standard procedures of the integral method [6] an ODE

$$\frac{dT_s}{dFo} = f_1(T_s, Fo), \quad 0 \leqslant Fo \leqslant Fo_\delta, \tag{4a}$$

subject to

$$T_s(0) = 1 \tag{4b}$$

is obtained.

As soon as $\delta = 1$ ($Fo = Fo_\delta$) the initial stage is terminated and equation (4) is no longer valid. In the second stage, the concept of penetration distance has no meaning and, consequently, only the natural boundary conditions (2b) and (2c) are applied. Now a quadratic profile which automatically satisfies the symmetry requirement imposed by boundary condition (2b) at $r = 0$ is assumed. Then, the integral method leads to an ODE

$$\frac{dT_s}{dFo} = f_2(T_s, Fo), \quad Fo_\delta \leqslant Fo \leqslant \infty, \tag{5a}$$

subject to

$$T_s(Fo_\delta) = T_{s,\delta}. \tag{5b}$$

3. DESCRIPTION OF THE APPROXIMATION FORMULAS

In an earlier paper [1], Kuo *et al.* did not apply the concept of penetration distance, but obtained a heat balance integral by integrating equation (1) from $r = 0$ to $r = 1$. In later papers (e.g. [2]), Kuo and Summerfield applied the procedure outlined above to a profile suggested by Lardner and Pohle [7]. Gough and Zwarts [3] considered the sphere as a plane semi-infinite body, using a result given by Goodman [6]. In the most simple case the heat flow to the particle is controlled by the convection resistance and therefore the particles can be assumed to behave spacewise isothermal. Thus, a heat balance on the particle, equating the change in heat capacity with the convective heat loss, easily yields an ODE for the particle temperature.

Since the comparative tests revealed that the formula presented by Kuo and Summerfield [2] achieves the best agreement with the numerical exact solution, the coefficients of that profile are given and the validity range of the formula is extended to times $Fo \geqslant Fo_\delta$.

Assuming a profile of the form

$$T = [C_0(Fo) + C_1(Fo)r + C_2(Fo)r^2 + C_3(Fo)r^3]/r \tag{6}$$

in ref. [2], the following ODE for T_s is derived:

$$\frac{dT_s}{dFo} = \left[\frac{12}{\delta}\{T_s(Fo) - 1 + Bi(Fo)[T_g(Fo) - T_s(Fo)]\} + \delta\left\{Bi(Fo)\frac{dT_g}{dFo} + [T_g(Fo) - T_s(Fo)]\frac{dBi}{dFo}\right\}\right] \times [6 + (Bi-1)\delta]^{-1} \quad \text{for } 0 \leqslant Fo \leqslant Fo_\delta \tag{7a}$$

subject to

$$T_s(0) = 1. \tag{7b}$$

Here δ is

$$\delta(Fo) = 3[T_s(Fo) - 1]/[T_s(Fo) - 1 + f(Fo)] \tag{8}$$

with f defined as

$$f(Fo) = Bi(Fo)[T_g(Fo) - T_s(Fo)]. \tag{9}$$

Equation (8) reveals that

$$\lim_{Bi \to 0} \delta = 3 > 1. \tag{10}$$

Mathematically, equation (10) means that equation (7a) can only be applied in cases where Bi exceeds a certain critical Biot number Bi_c, since $\delta(Fo)$ is only allowed to vary within $[0, 1]$. As a physical consequence of equation (10) there is no sense in applying approximation formulas like equation (7) to cases where the conduction resistance of the body is small compared with the convection resistance and therefore, as is well known, for $Bi \to 0$ the assumption of a spatial uniform pellet temperature is justified. Apart from this, equation (7) has a singularity for $\delta \to 0$. For practical purposes this singularity can be suppressed by setting $\delta(0) = \varepsilon$ where ε is any small number, e.g. $\varepsilon = 0.001$.

The Fo-dependent coefficients in equation (6) can be written as

$$\begin{aligned} C_0 &= -(1-\delta)^3 f \delta^{-1}(3-\delta)^{-1}, \\ C_1 &= -3C_0/(1-\delta) + 1, \\ C_2 &= 3C_0/(1-\delta)^2, \\ C_3 &= -C_0/(1-\delta)^3. \end{aligned} \tag{11}$$

For $Fo \geqslant Fo_\delta$ the application of the integral method to the quadratic profile

$$T = A_0(Fo) + A_1(Fo)r^2 \tag{12}$$

yields

$$\frac{dT_s}{dFo} = \left\{12\,Bi(Fo)[T_g(Fo) - T_s(Fo)] + Bi\frac{dT_g}{dFo} + (T_g - T_s)\frac{dBi}{dFo}\right\} \times [4 + Bi(Fo)]^{-1} \quad \text{for } Fo \geqslant Fo_\delta \tag{13a}$$

subject to

$$T_s(Fo_\delta) = T_{s,\delta} \tag{13b}$$

and

$$\begin{aligned} A_0 &= T_s - 0.5f, \\ A_1 &= 0.5f \end{aligned} \tag{14}$$

where f is defined in equation (9). It can be easily checked that for $Fo = Fo_\delta$ equation (6) and equation (12) represent identical profiles, although they are only linked by the initial condition equation (13b).

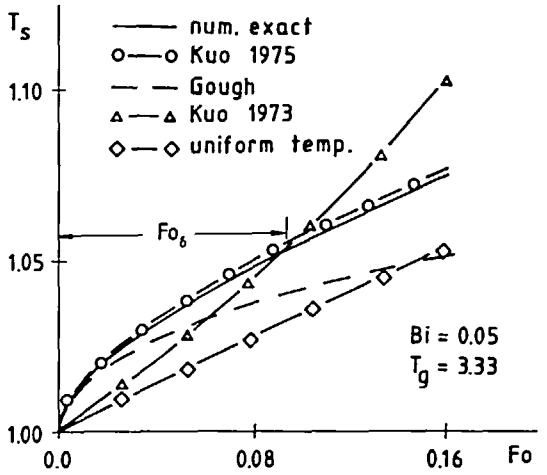

FIG. 1. The different approximations and the numerical exact solution for short time intervals.

4. CLOSED FORM SOLUTIONS

For constant values of T_g and Bi the following solutions of equations (7) and (13) are obtained

$$Fo = \frac{0.75}{(1-Bi)^2}\left\{\ln\left(\frac{\vartheta}{Bi}\right) - \frac{1}{2}\left[1-\left(\frac{Bi}{\vartheta}\right)^2\right]\right\}$$

for (15)

$$1 \geqslant \vartheta \geqslant \frac{2}{2+Bi}, \quad 0 \leqslant Fo \leqslant Fo_\delta,$$

and

$$Fo = Fo_\delta + \frac{4+Bi}{12\,Bi}\ln\left[\frac{2}{(2+Bi)\vartheta}\right]$$

for (16)

$$\frac{2}{2+Bi} \geqslant \vartheta \geqslant 0, \quad Fo_\delta \leqslant Fo \leqslant \infty,$$

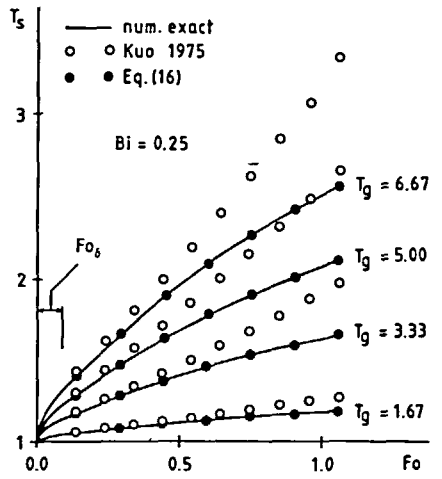

FIG. 2. Solution of equations (15) and (16) and numerical exact solution for different values of T_g.

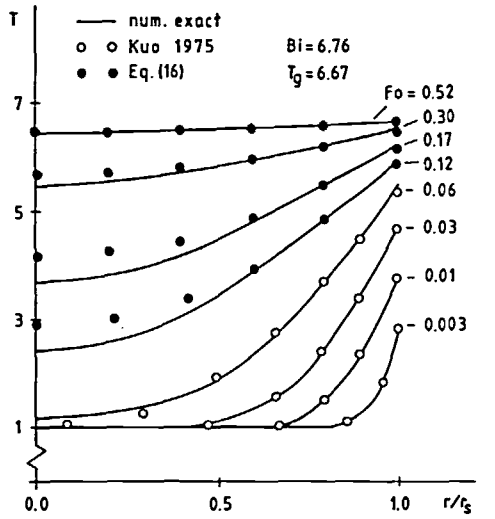

FIG. 3. Development of temperature profiles with time inside a particle.

where

$$Fo_\delta = \frac{0.75}{(1-Bi)^2}\left[\ln\left(\frac{3}{2+Bi}\right) + \frac{Bi(Bi+4)-5}{18}\right]. \quad (17)$$

5. RESULTS AND DISCUSSION

The results represented in Figs. 1–3 refer to constant values of T_g and Bi. In Fig. 4 both T_g and Bi are assumed to be exponential functions of time. In all cases considered the value of T_0 is $T_0 = 300$ K.

In Fig. 1, the solutions of the different approximation formulas are plotted for an Al_2O_3-particle with $r_s = 1$ mm. The corresponding time interval is $t = 120$ ms. Although exceeding its theoretical validity range $[0, Fo_\delta]$, the solution obtained by Kuo and Summerfield [2] achieves best agreement with the numerical exact solution. For short time intervals the formula of Gough and Zwarts [3] yields acceptable results too. However, the deviations from the exact solution increase with increasing time. This is to be expected since in the case of a planar semi-infinite body the volume into

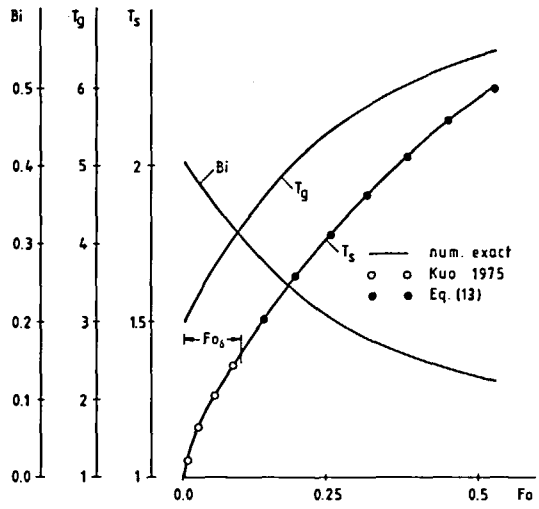

FIG. 4. Solution of equations (7) and (13) and numerical exact solution for exponentially varying gas temperature and Biot number.

which the heat diffuses remains the same for equal spatial increments. The solution of Kuo *et al.* [1] yields too low temperatures during the initial stage, whereas it overshoots the exact solution when it is extended to larger time intervals. Both effects are probably due to a not appropriately chosen approximation formula. The considerable difference between the numerical solution and the solution obtained under the assumption of a spatial uniform temperature is to be expected for the used value of Bi.

In Fig. 2 results are represented for an Al_2O_3-particle with $r_s = 5$ mm, obtained from the approximate solutions equations (15) and (16), and the numerical solution. The corresponding time interval is $t = 20$ s, which exceeds the validity range of equation (15) by a factor of 25. The parameter varied is T_g. Again, the solution of Kuo and Summerfield [2] achieves excellent agreement with the exact solution as long as it does not exceed its validity range too far. However, the deviation from the numerical solution increases with increasing T_g and increasing Fo, whereas the solutions obtained from equation (16) yield good agreement with the numerical solution for $Fo \geqslant Fo_\delta$.

Figure 3 shows the temperature profiles inside a solid propellant-particle with $r_s = 5$ mm. The corresponding time interval is 1.7 s. The good agreement with the numerical solution during the initial stage slowly decreases with increasing penetration of the thermal wave. Although equation (6) and equation (12) provide identical results for $Fo = Fo_\delta$, there occurs a rapid displacement of the temperature profile in the environment of the center shortly after the beginning of the second stage. This deviation decreases with increasing time while in the outer layers beneath the particle surface there is always good agreement with the numerical solution.

Figure 4 again shows results for an Al_2O_3-particle with $r_s = 5$ mm. The corresponding time interval is $t = 10$ s. In this case, T_g and Bi are exponential functions of Fo. For $0 \leqslant Fo \leqslant Fo_\delta$ the approximate solutions are obtained from equation (7) while for $Fo \geqslant Fo_\delta$ equation (13) is solved. These approximate solutions, the numerical solution, T_g, and Bi are presented. It can be gathered that there is again good agreement between the numerical solution and the approximate solution assembled from equations (7) and (13).

6. CONCLUDING REMARKS

The approximation formula given by Kuo and Summerfield [2] turned out to yield the best agreement to the numerical solution, however, its validity range is limited. For that reason an extended solution, linked to that formula, was derived. For the case that both T_g and Bi are constants, the corresponding ODE's could be solved analytically. The surface temperature obtained by these two approximation formulas agrees very well with the numerical solution, whereas in the center of a pellet deviations can occur.

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ON THE EFFECT OF A LARGE TEMPERATURE DIFFERENCE ON THE VELOCITY AND TEMPERATURE PROFILES FOR THE TURBULENT FLOW OF AIR IN A TUBE

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NOMENCLATURE

- c_p specific heat of the gas at constant pressure [J kg^{-1} K^{-1}]
- c_{px} specific heat of the gas at constant pressure evaluated at T_x [J kg^{-1} K^{-1}]
- c_{pw} specific heat of the gas at constant pressure evaluated at the wall temperature T_w [J kg^{-1} K^{-1}]
- q_w heat flux to or from the gas at the wall [W m^{-2}]
- T gas temperature [K]
- T_B gas bulk temperature [K]
- T_w wall temperature [K]
- T_x geometrical mean between wall and gas bulk temperature [K]
- t_w^+ dimensionless gas temperature with gas properties evaluated at T_w
- t_x^+ dimensionless gas temperature with the gas properties evaluated at T_x
- u_w^* friction velocity with the gas density evaluated at T_w [m s^{-1}]
- u_x^* friction velocity with the gas density evaluated at T_x [m s^{-1}]
- u_w^+ dimensionless gas velocity with the gas density evaluated at T_w
- u_x^+ dimensionless gas velocity with the gas density evaluated at T_x
- y radial distance from the wall of the considered point [m]